COM2031 Advanced Algorithms, Autumn Semester 2019

Lab 9: Reductions: More Applications of Network Flow

SOLUTIONS

Purpose of the lab

This week's lab is to explore two problems covered in Topic 4 (Network Flow Applications): **Project Selection**, and **Survey Design**.

For each of these, you will reduce instances of these problems to instances of a Max-Flow/Min-Cut problem. That problem can then be solved with the Ford-Fulkerson algorithm. You will then need to translate the solution back into a solution to the original problem.

In order to best achieve an understanding of what is going on, you should use the tool at:

https://www-m9.ma.tum.de/graph-algorithms/flow-ford-fulkerson/index_en.html

This allows you to build flow networks, and to apply Ford Fulkerson to find the Max Flow / Min Cut. You will translate the problem given into the appropriate flow network, run the algorithm, and in that way obtain the solution.

If you wish to run code to find the solutions then you may also do this, but this is an optional extra. The key element of the lab is to use the visualisation to understand the reduction of the other problems to Max Flow/Min Cut. An implementation of the Ford Fulkerson algorithm using adjacency matrices is provided in the solutions of lab 7 in SurreyLearn. Or you can use your own one that you developed in Lab 7 if you prefer.

1. Project Selection

The project selection problem is as follows (from Lectures) :

There is a set P of possible projects. Each project v has associated revenue p_v.
Some projects generate money, in which case ($p_v > 0$). Others cost money ($p_v < 0$)
Each project v has a set of prerequisites – other projects that must be done in order to do project v.
A subset of projects $A \subseteq P$ is feasible if the prerequisite of every project in A also belongs to A.

Project selection: Find a maximal project selection that satisfies all dependency requirements and maximises net total profit.

So a Project Selection problem gives a list of projects, and for each project there is a profit or cost, and a set of other projects that it is dependent on.

Exercise 1a: Here is a set of 4 projects. Project A gives a profit of 6 and is dependent on project C; project B gives a profit of 4 and is dependent on both C and D. Project C has a cost of 3, and project D has a cost of 5.

This means that project A by itself is not an acceptable selection, since if A is included then C must also be included since A is dependent on C.

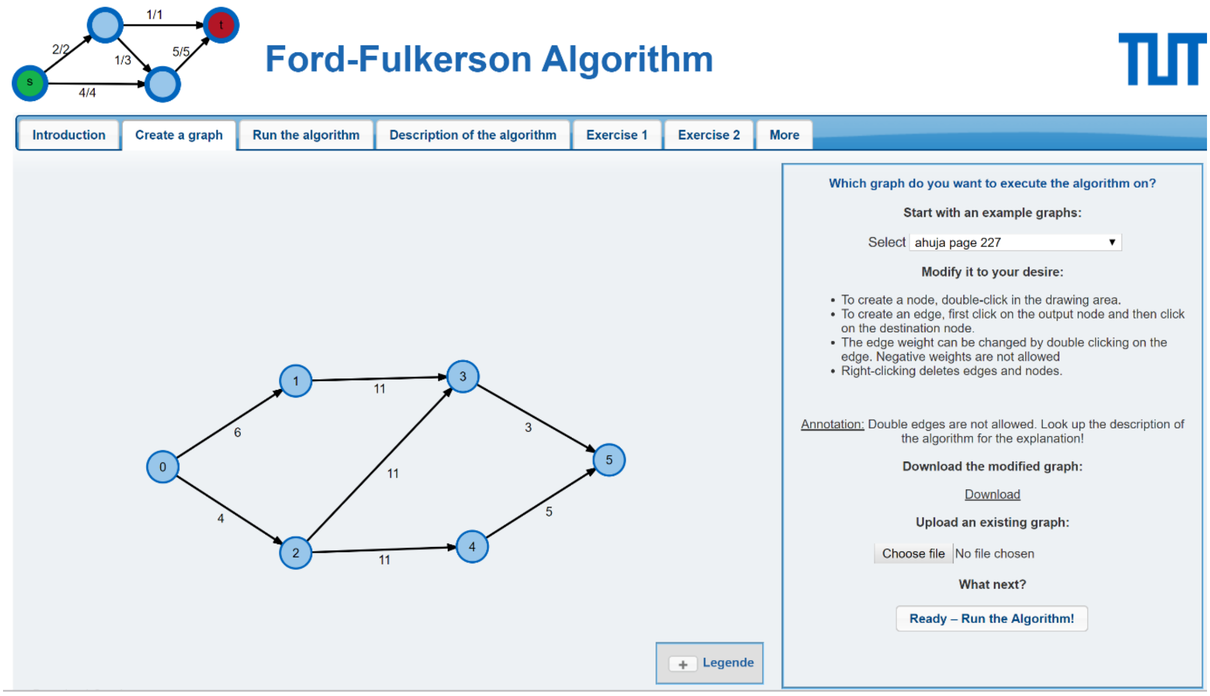
On the other hand, the set {A, B, C, D} is an acceptable selection, since all of the dependencies of A and B are also included. This selection gives an overall profit of 2.

Project	p_v	Dependencies
A	6	C
B	4	C, D
C	-3	
D	-5	

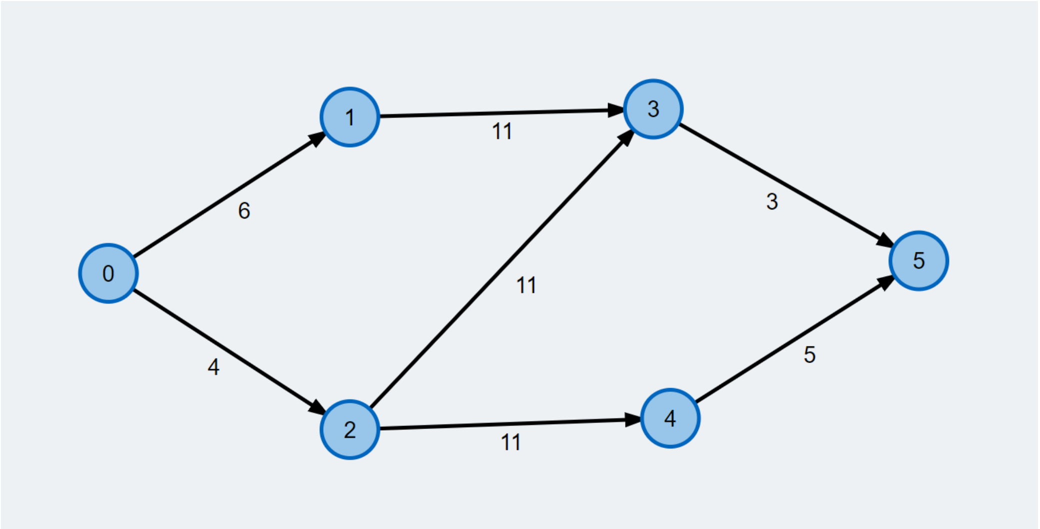
In order to convert this Project Selection problem into a Network Flow problem we create the flow graph with the following nodes and edges:

- A source node s and a sink node t
- A node for each project
- For each project with cost $p_v < 0$ add an edge with weight - p_v from the associated node to the sink t.
- For each project with positive profit $p_v > 0$ add an edge from the source s to the corresponding node with weight p_v.
- For each dependency between projects add an edge from a project node to any node on which it is dependent. In the lectures these edges are labelled with infinity, but the key point is that they should have so much capacity that they are not full even with the maximum flow. We can see that the sum of the weights from s is a total of 10, so the maximum flow can be no more than 10. Therefore if we assign a weight of 11 to each of these edges then there is no flow in which they will be full

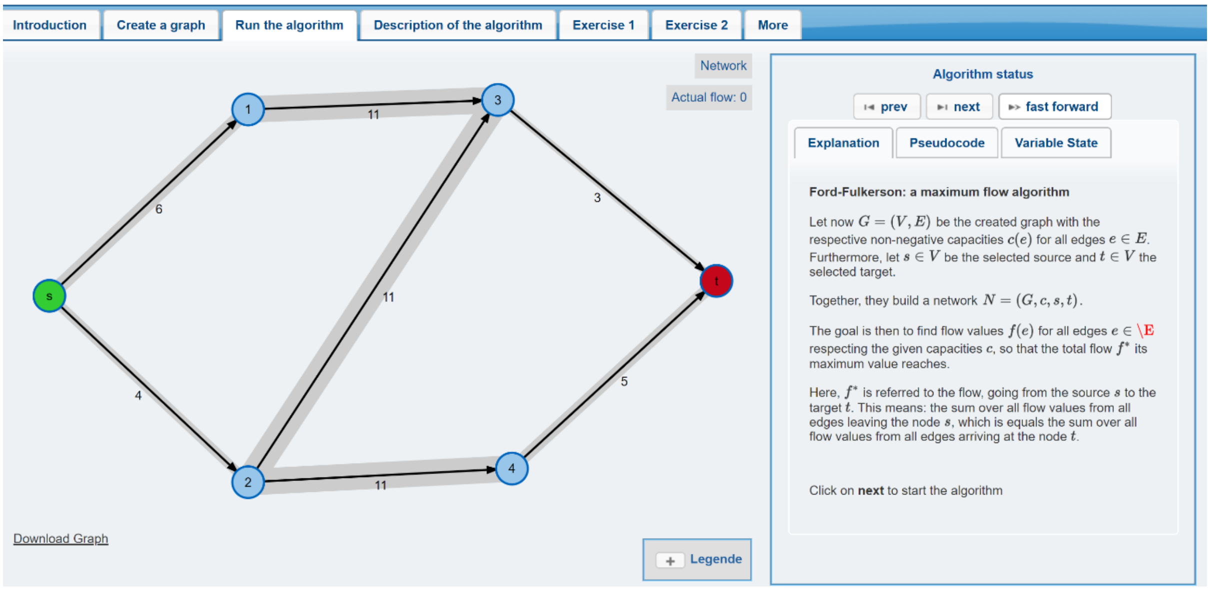
Create the associated graph in the tool. You should end up with something like the following on your screen (though possibly with different numbers on the nodes)



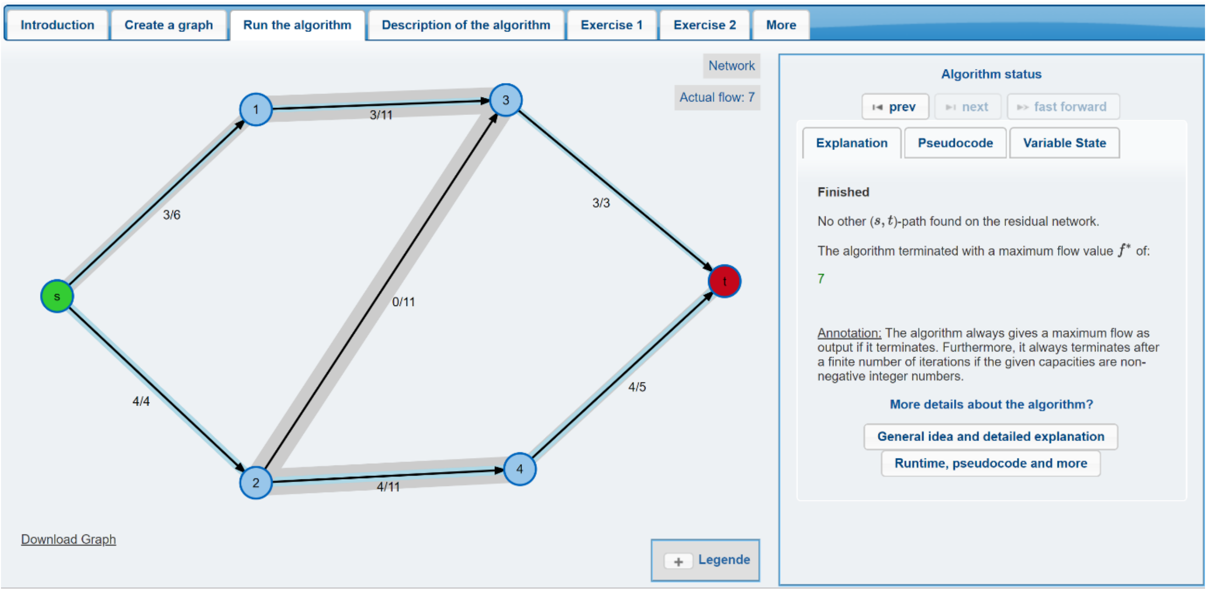
The graph in close up is as follows:



Now select the “run the algorithm” tab, and select the appropriate nodes for s and t:

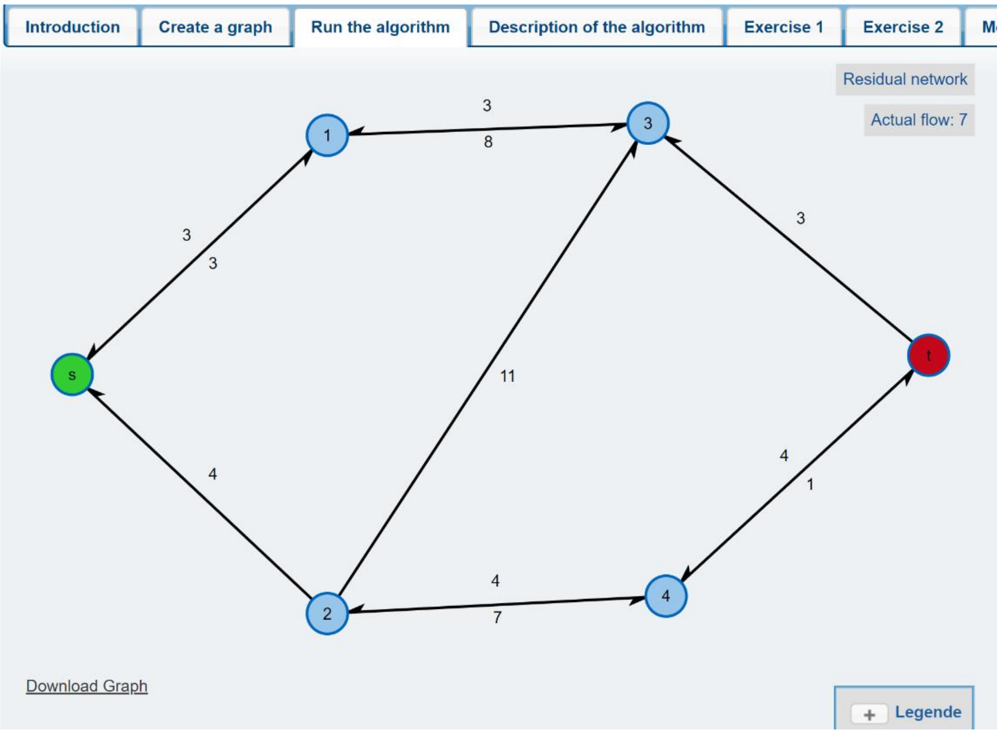


Run the algorithm through to the end. You can select "fast forward" or you can step through the algorithm to watch it build up the maximum flow. After you have run the algorithm you should reach the following, which gives the maximum flow.

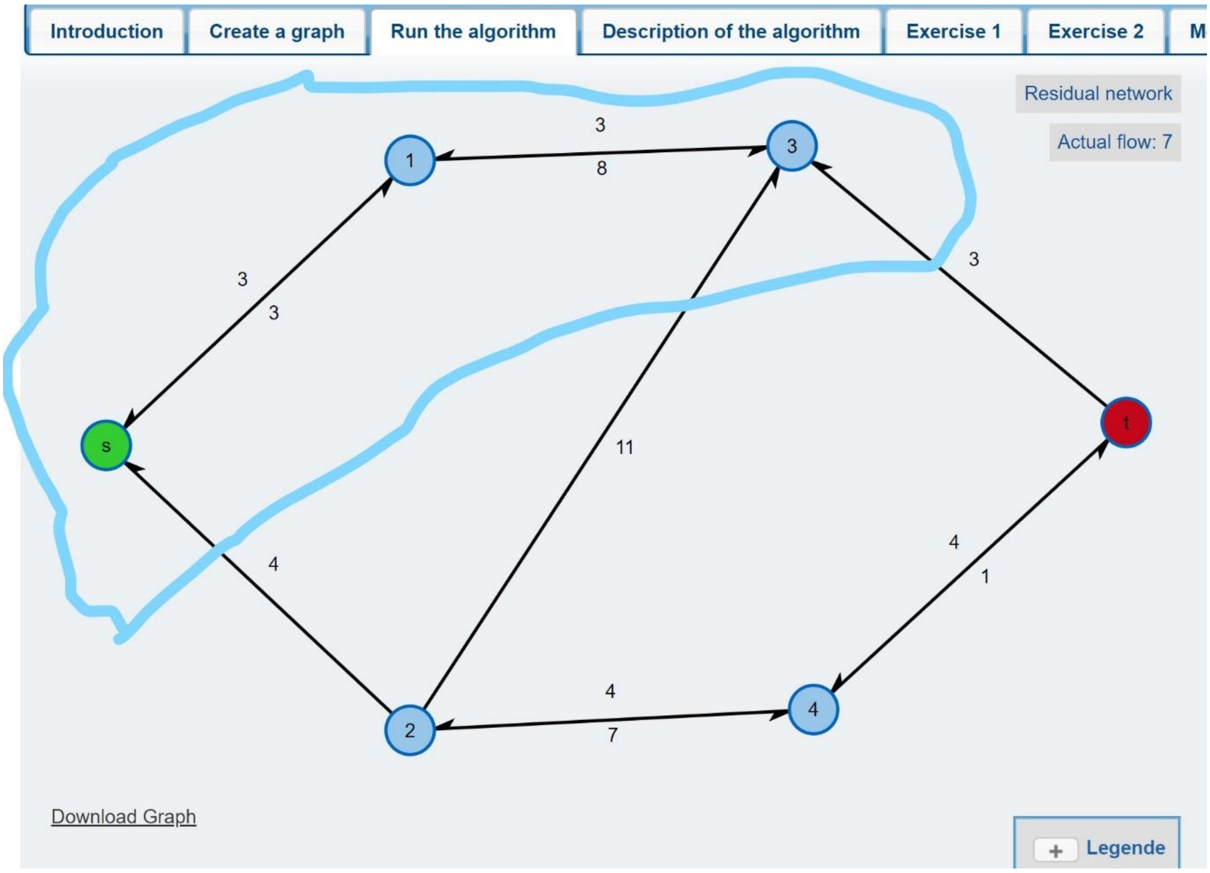


We need to find the minimum cut corresponding to the maximum flow. The set of projects to select will be those that are connected to the source s in the minimum cut.

To find the minimum cut we need to find the residual graph for this maximal flow. To do this you should select the "prev" tab twice to go back two steps in the algorithm. This yields the following residual graph:



Many of the edges have capacity in both directions but some do not. Here there is an edge from s to A (node 1), and from A to C (node 3). However no other nodes are reachable from s, 1 or 3. Therefore this is the min cut set:



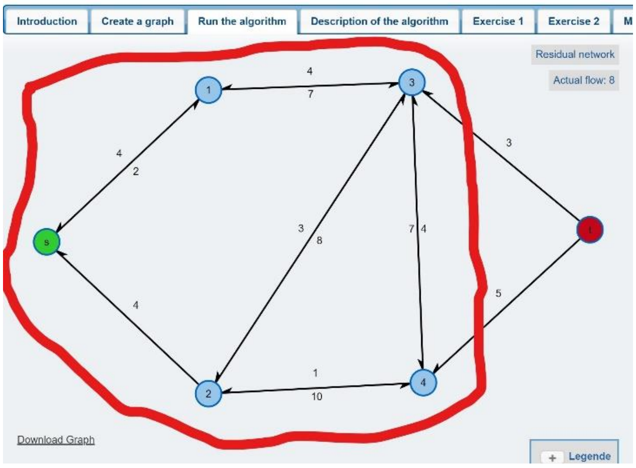
The projects in the min cut connected to s are the projects A and C, so the selection of projects giving the maximum profit is {A, C}.

Exercise 1.b:

Now work out the best set of projects to select for the following set of projects and dependencies:

Project	p_v	Dependencies
A	6	C
B	4	C, D
C	-3	D
D	-5	

Solution

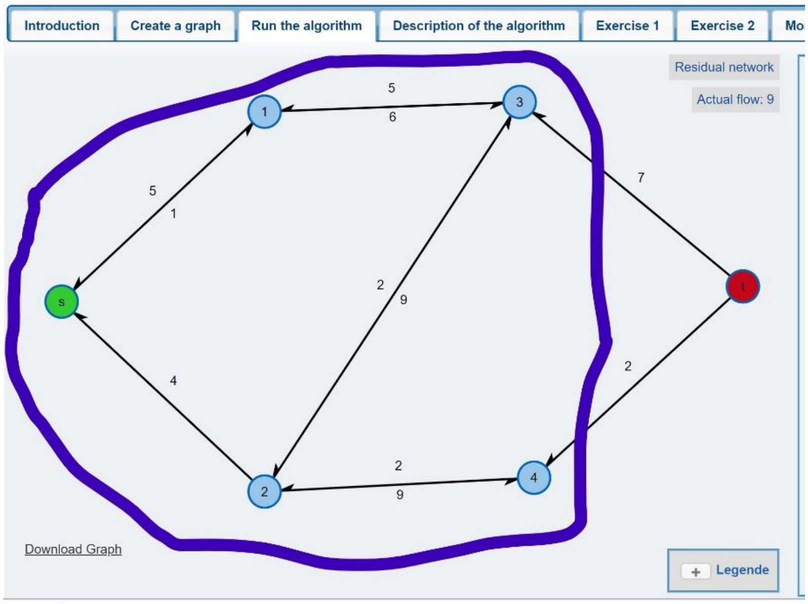


Here is the min cut for the appropriate graph, so the projects to select are {A,B,C,D}: all of them.

Exercise 1.c:

Project	p_v	Dependencies
A	6	C
B	4	C, D
C	-7	
D	-2	

Solution:

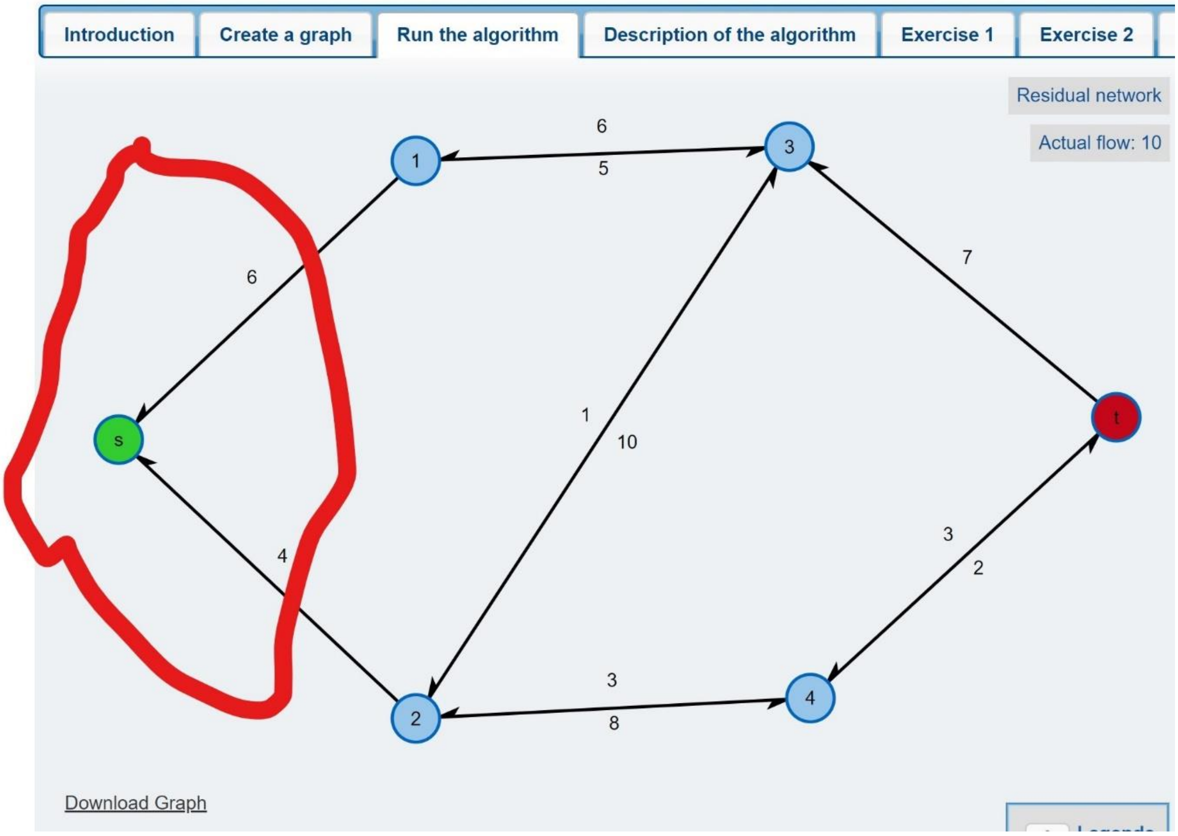


Again the set of projects to select is {A, B, C, D}

Exercise 1.d:

Project	p_v	Dependencies
A	6	C
B	4	C, D
C	-7	
D	-5	

Here the solution from the residual graph is as follows:



The best set of projects to select in this case is none at all. No nodes are reachable from s in the residual graph.

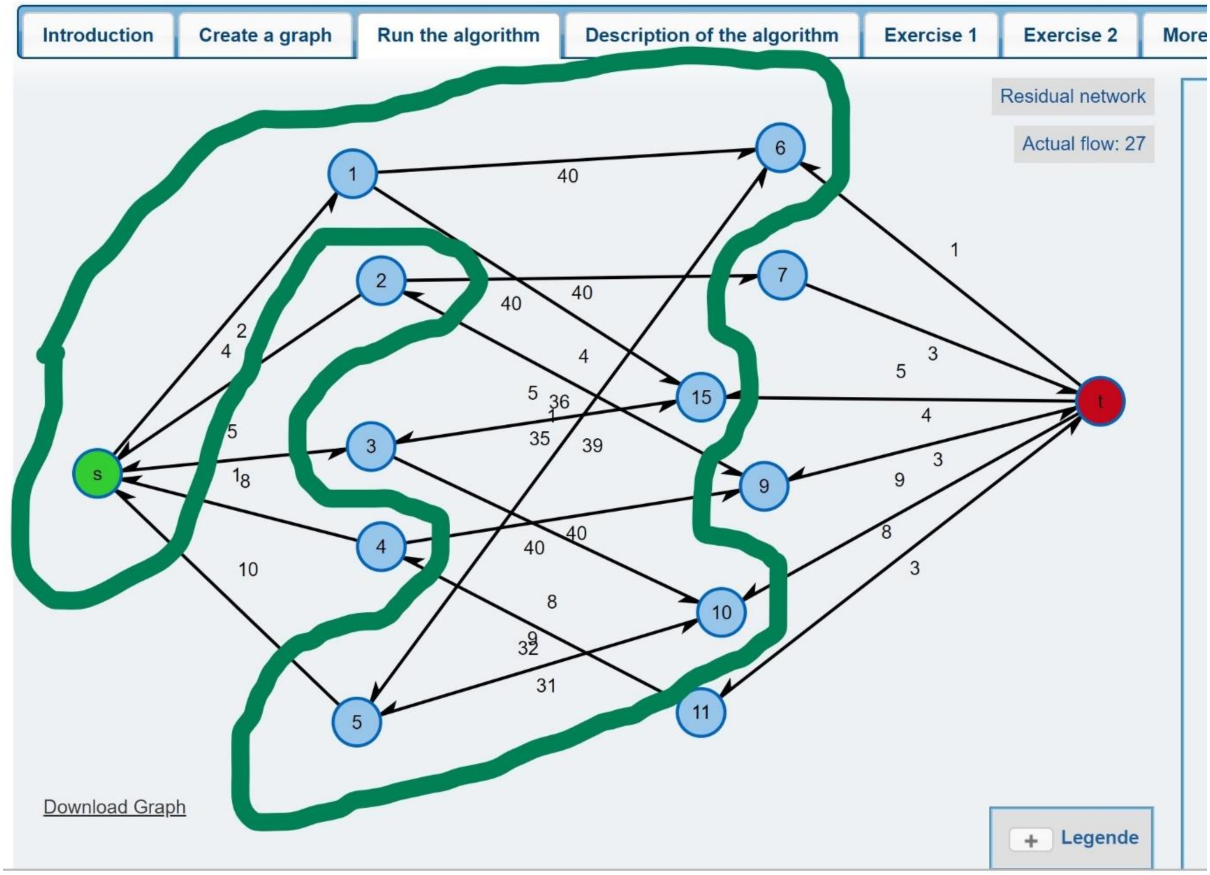
Exercise 1e:

Now consider a larger set of projects, with more dependencies as follows:

Project	Profit/cost	Dependencies
A	2	F, H
B	4	G, I
C	6	H, J
D	8	I, K
E	10	F, J
F	-1	
G	-3	
H	-5	
I	-7	
J	-9	
K	-11	

What is the set of projects with the maximum profit?

The min cut given by the residual graph is as follows:

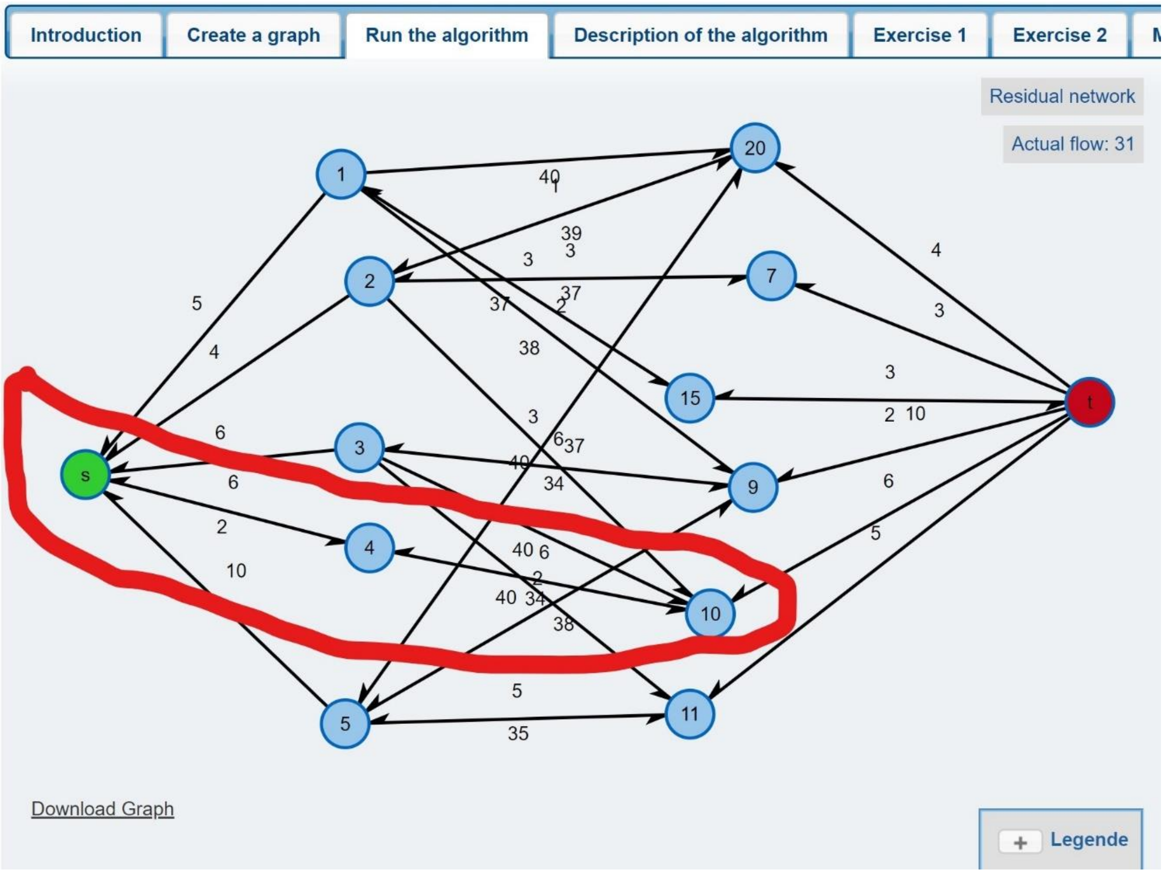


This gives the optimal project selection as {A, C, E, F, H, J} with a profit of 3.

Exercise 1f:

Project	Profit/cost	Dependencies
A	5	F, H, I
B	4	F, G, J
C	6	I, J, K
D	8	J
E	10	F, I, K
F	-4	
G	-3	
H	-5	
I	-10	
J	-6	
K	-5	

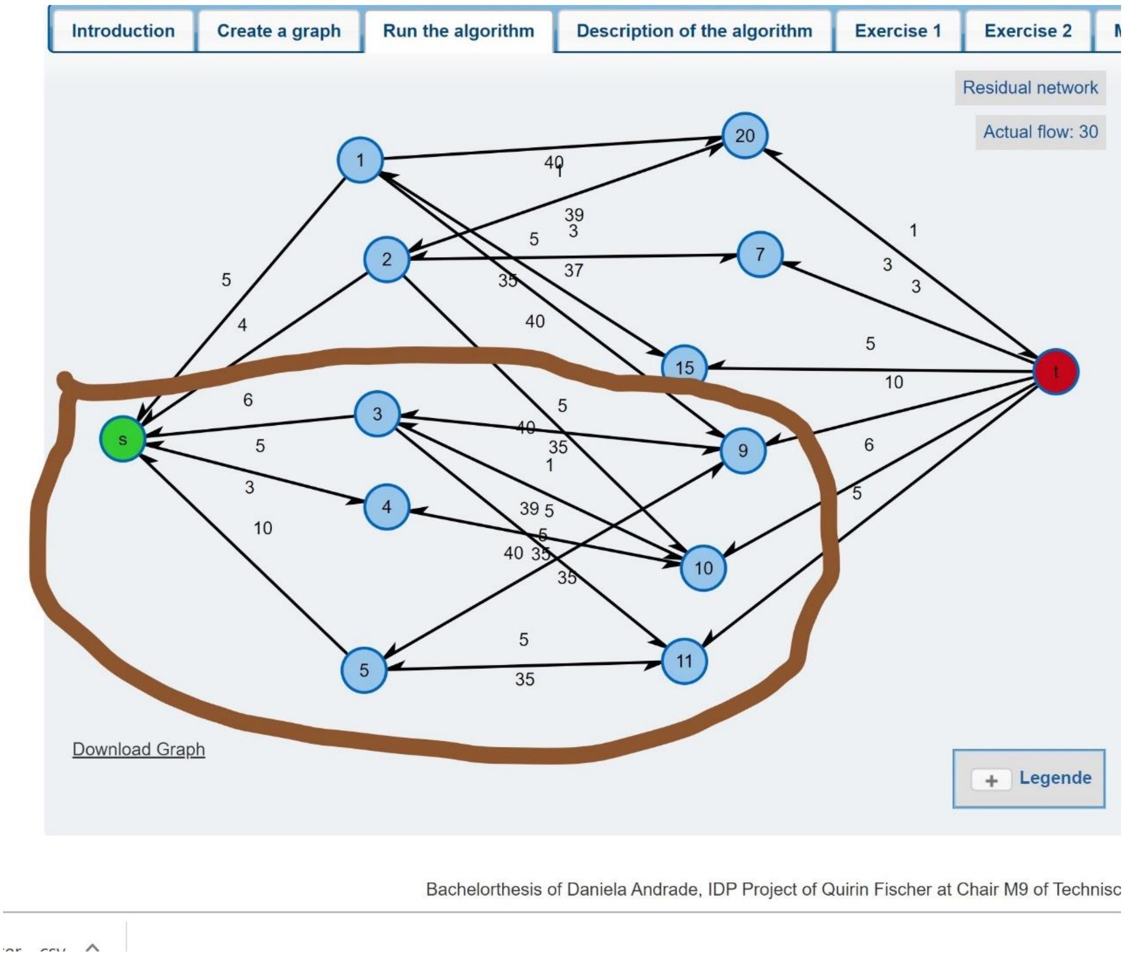
The optimal project selection set is given by the residual graph as follows:



This is the set {D, J} with a profit of 2.

Exercise 1g: The same projects and dependencies as exercise 1f except remove the dependency from E to F.

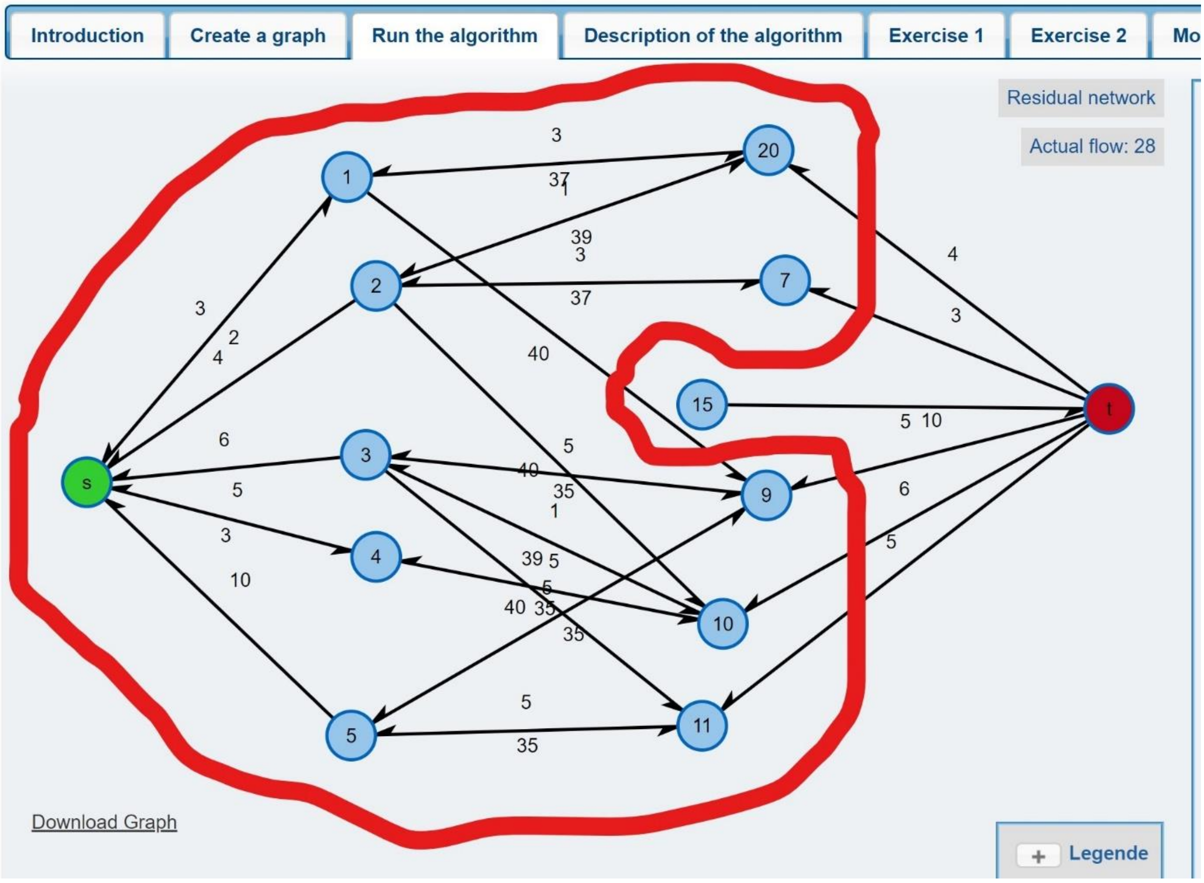
Solution: The residual graph provides {C, D, E, I, J, K} in the minimal cut, as pictured below. This provides a profit of 3.



Exercise 1h: Find one dependency to remove from the problem of 1g so that the optimal solution has all of the profit making projects (i.e. contains all of A, B, C, D, E). Verify that your answer is correct by computing the min cut.

Solution: remove the dependency from A to H. Intuition: A is the only project dependent on H, so removing this dependency means that the optimal solution need not contain H even if it contains all of A to E.

Illustrated below:



- Example Printing of the min cut edges can be found in the printMinCut method in the fordFulkerson class provided to you in Lab 7 Solutions. The project Selection method require tracing from s to all reachable vertices in the first set of the cut, which is slightly different than printing the min-cut edges themselves.

2. Survey Design (slides 64 – 68)

The Survey Design problem. We have a bipartite graph with a number of links from one set L to another set R: Each node in L and each node in R has a constraint on the number of links to be selected: a range, from a lower bound, to an upper bound. We need to select a collection links so that each node has a number of edges within its range.

One example is links between customers and products thjat they have purchased. In that case the problem is formulated as follows:

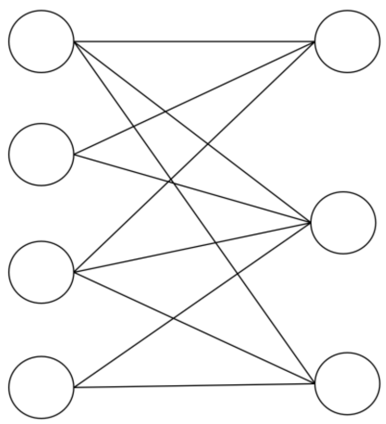
We want to select a collection of edges so that

- Design survey asking n_1 consumers about n_2 products.
- Can only survey consumer i about a product j if they own it.
- Ask consumer i between c_i and c_i' questions.
- Ask between p_j and p_j' consumers about product j.

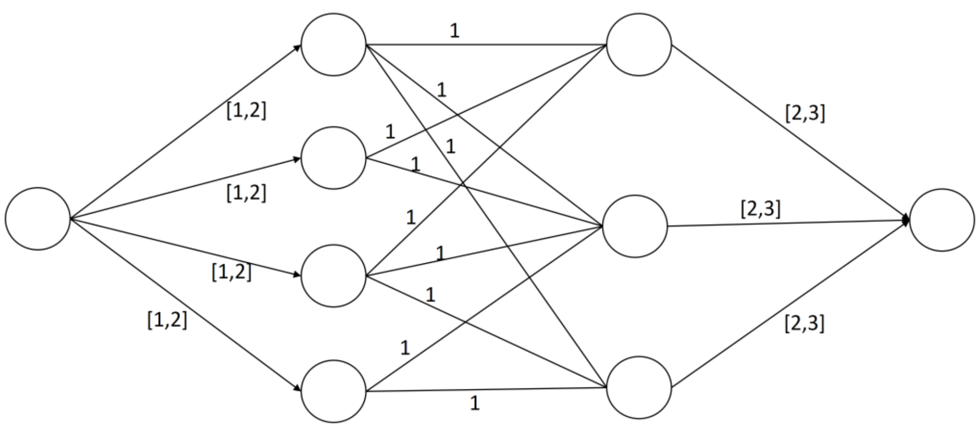
Goal. Design a survey that meets these specs, if possible.

This can be reduced to a Circulation with Demand problem, which can then be reduced to a Network Flow problem and fed into the tool, as described in the lectures. The first exercise will take you through the series of steps. The following exercise will just pose the question for you to solve and you have to follow the same steps yourself.

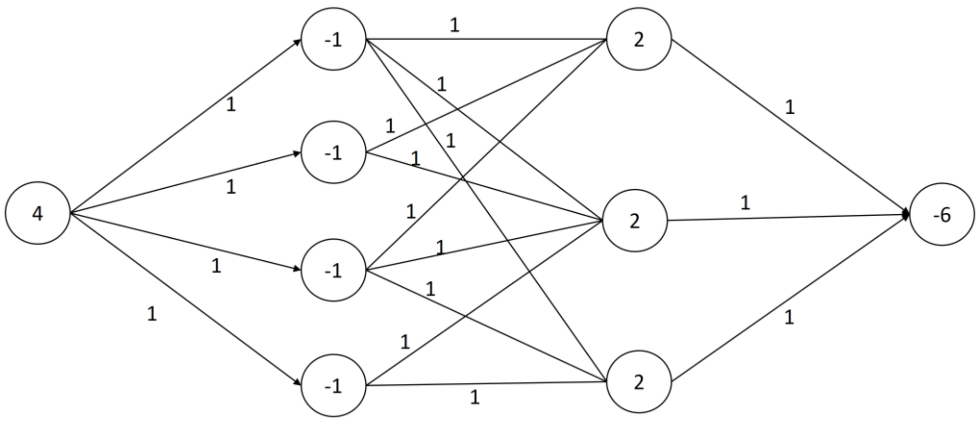
Here we have a bipartite graph:



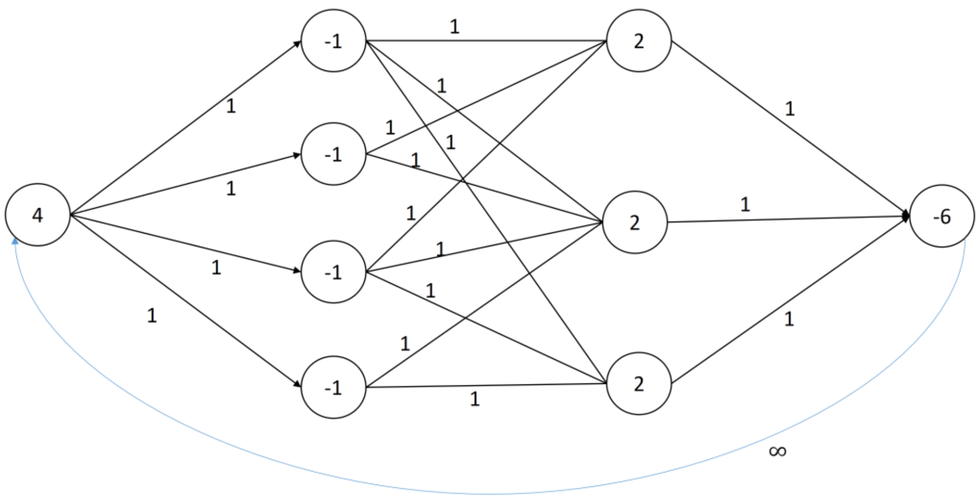
We will introduce the requirement that every node on the left must have between 1 and 2 links, and every node on the right must have between 2 and 3 links. These are incorporated by introducing s and t with the ranges labelling the edges. The original edges are all labelled with capacity 1:



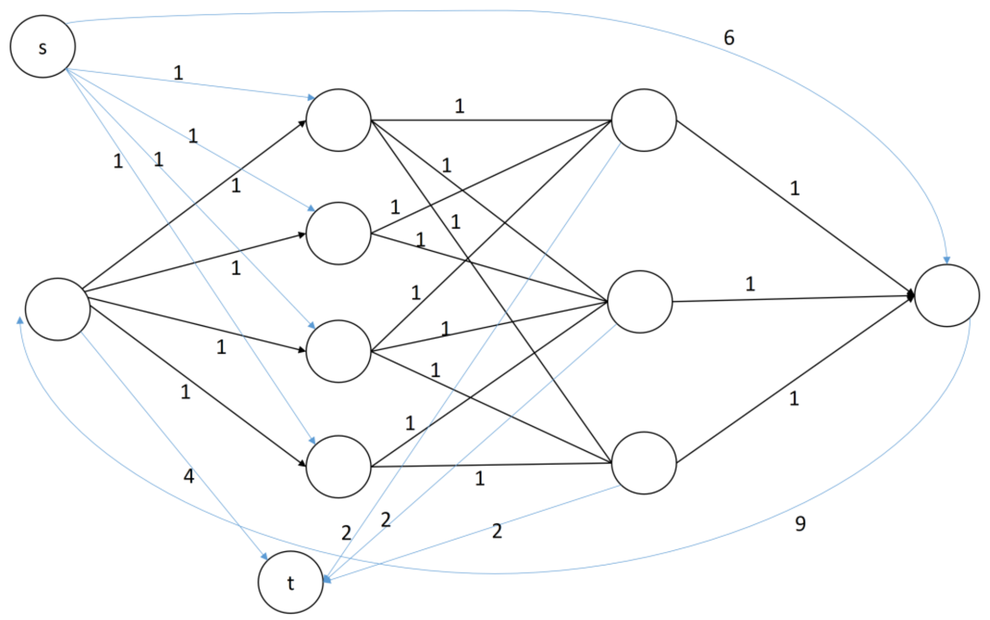
This can now be reduced to a Circulation with Demands problem (see slides 58-59) by initially labelling each node with demand 0, and then adjusting the demands according to the lower bound on the edge, as explained in slide 59. This results in the following graph:



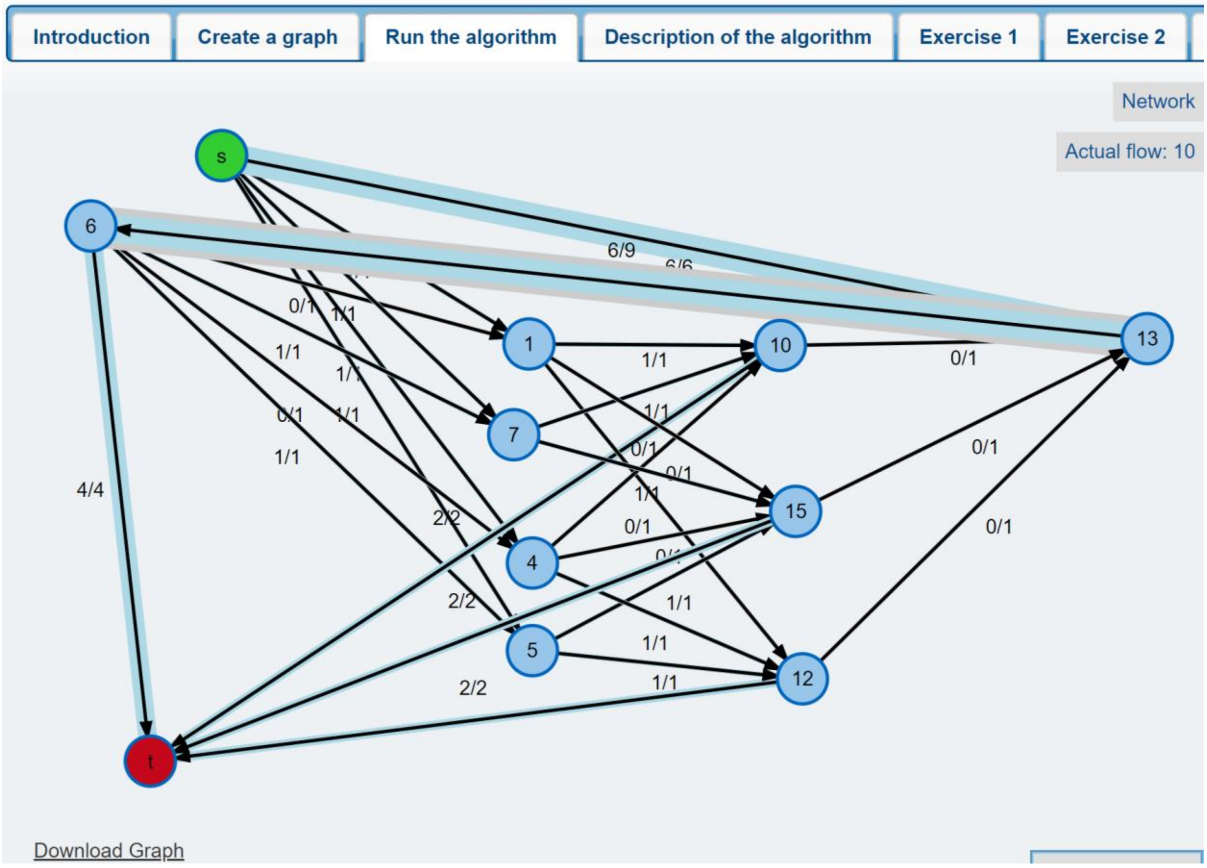
For technical reasons we need to connect the sink t to the source s (with infinite capacity) to complete the circulation:



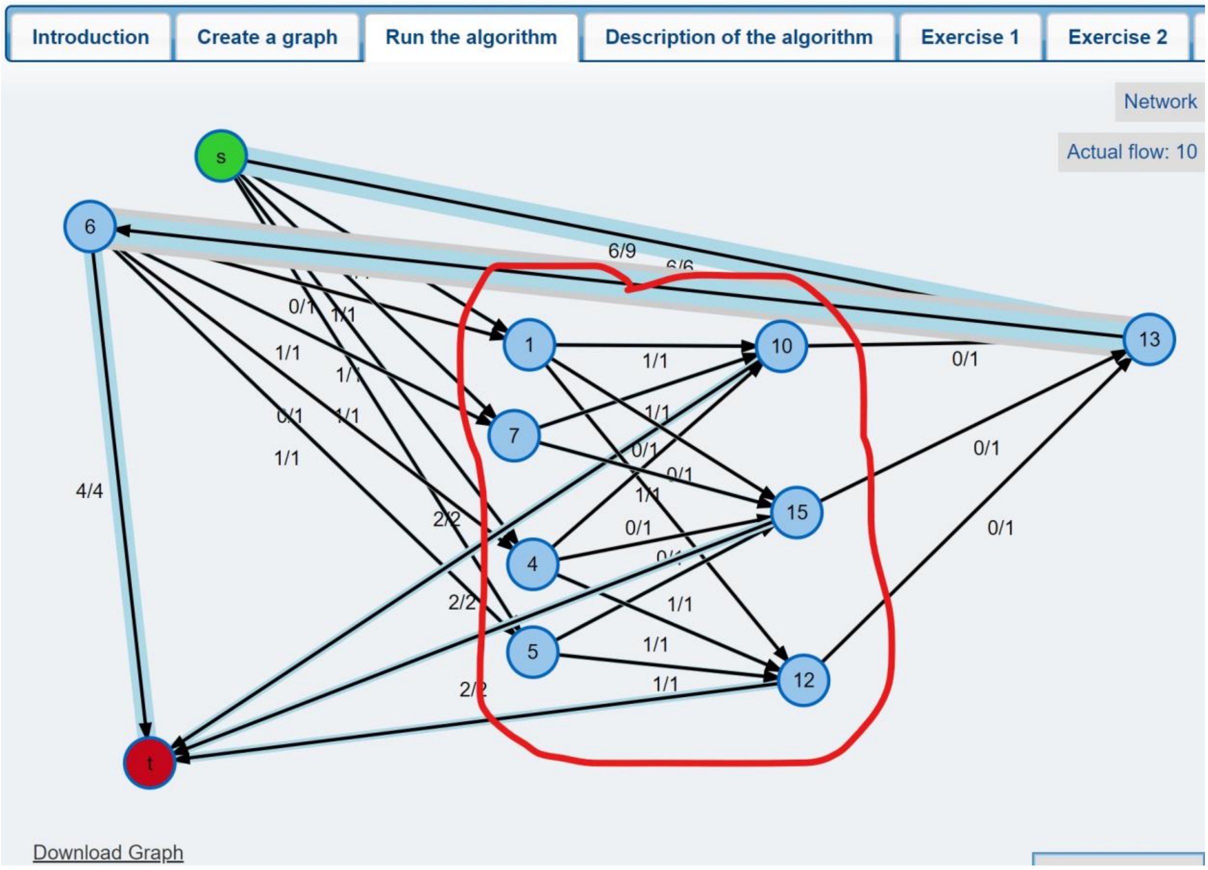
This is now a circulation with demands problem, so it can be reduced to a network flow problem by introducing a new source and sink and connecting the source to nodes with supply (negative values), and the sink with nodes with demand (positive values), as explained in slides 46-49:



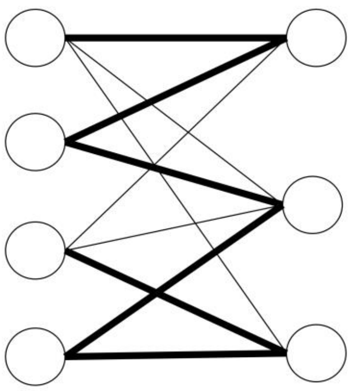
This can now be run through the tool to find the maximum flow through the network. This yields the following:



The flows along the edges between the nodes in L and the nodes in R give the result to the original survey design question: where there is a flow of 1 then the edge should be included, and where there is a flow of 0 the edge should not be included. This is the region to look at:



Hence the solution to the original Survey Design problem is as follows

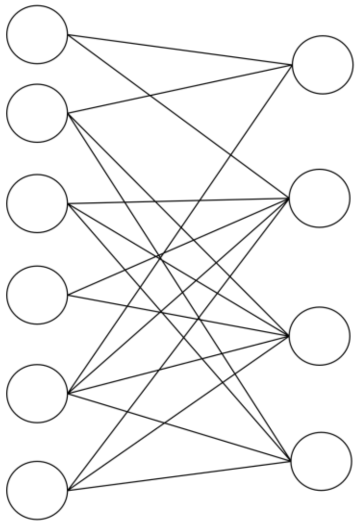


The original constraints are met: for the set of selected edges, each node on the left has 1 or 2 edges, and each node on the right has 2 edges.

Exercise 2b:

Here is a bipartite graph. We wish to select a set of edges so that every node on the left has between 1 and 3 edges (range [1,3]); and every node on the right has between 3 and 4 edges (range [3,4]).

Apply the method of the previous question to find such a selection.



Solution:

There are numerous solutions. Here is one: